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Proof of Dirac's conjecture concerning the generators of surfaces of physically equivalent points

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Abstract. Dirac conjectured that in a constrained Hamiltonian system the surfaces of points physically equivalent to one another are generated by the totality of first-class constraints, both primary and secondary. A proof is given.

1. Introduction

In his book Dirac (1964) postulates, but does not prove, that in a constrained Hamiltonian system surfaces of points physically equivalent to one another are generated by the set of all first-class constraints, both primary and secondary. I supply the proof for systems having a finite number of degrees of freedom, subject to the following assumptions:

(1) Every point in phase space has a neighbourhood within which all solutions to the equations of motion are analytic functions of time, provided the arbitrary functions are themselves analytic.

(2) The ranks of various matrices I shall consider do not vary over the physically allowed surface in phase space.

(3) The surface of points physically equivalent to some particular point is not enlarged by considering trajectories determined by non-analytic arbitrary functions, nor by considering trajectories which start outside the neighbourhood defined in assumption (1).

The proof will be in two parts. In the first I shall show that every first-class constraint is a linear combination of quantities of the form

$$H^r\phi$$

where H is the Hamiltonian, ϕ is a first-class primary constraint, r is some integer, and where I have employed the notation

$$A_1A_2\ldots A_r \overset{\text{def}}{=} \{A_1, \{A_2, \ldots \{A_{r-1}, A_r\} \ldots\}\}.$$

If, and only if, there is the possibility of confusion, I shall denote ordinary, arithmetic, multiplication by a dot:

$$A_1 \cdot A_2 \overset{\text{def}}{=} A_1 \times A_2.$$

In the second part, I shall show that such quantities also constitute the complete set of generators of the surfaces of physically equivalent points.

2. The structure of the system of constraints

In the Dirac formulation of classical mechanics there is associated to each of a large class of Lagrangians, a Hamiltonian formalism in which the system is constrained to lie on a surface in phase space defined by the vanishing of the primary constraints. The equations of motion are generated by the expression

$$H_0 + u_a \cdot \phi_a$$

where H_0 is the Hamiltonian, the ϕ_a are the primary constraints, and the u_a are arbitrary functions of position in phase space. If the system admits cyclic motion, they are arbitrary functions of time also.

It may happen that some trajectories leave the physical surface. To ensure consistency, it is then necessary either to reduce the arbitrariness in the equations of motion (by fixing some of the u), or to reduce the dimensionality of the physically allowed surface (by imposing further, secondary, constraints), or both. The structure of the system of constraints after consistency has been imposed is described by the following:

Theorem 1. There exists a linear, invertible, transformation of the original set of primary constraints. The new primary constraints defined by this transformation (and which determine the same surface as the old ones) may be sorted into r sets of primary second-class constraints, for some integer r, and one set of primary first-class constraints. A new Hamiltonian H may be formed from the old Hamiltonian H_0 by adding to it a suitable linear combination of second-class primary constraints. H is first-class. The secondary constraints are then formed by taking Poisson brackets between H and the primary constraints as indicated in table 1. The second-class secondary constraints so obtained are all linearly independent. The first-class secondary constraints may not be. This tabulation of constraints is exhaustive, however.

Table 1.

	Second-class constraints							First-class constraints
Primary constraints	$\phi_a^{(1)}$	$\phi_a^{(2)}$	$\phi_a^{(3)}$	$\dots$	$\phi_a^{(r-2)}$	$\phi_a^{(r-1)}$	$\phi_a^{(r)}$	ϕ_a^{F}
Secondary constraints		$H\phi_a^{(2)}$	$H\phi_a^{(3)}$	$\dots$	$H\phi_a^{(r-2)}$	$H\phi_a^{(r-1)}$	$H\phi_a^{(r)}$	$H\phi_a^{\mathrm{F}}$
			$H^2\phi_a^{(3)}$	$\dots$	$H^2\phi_a^{(r-2)}$	$H^2\phi_a^{(r-1)}$	$H^2\phi_a^{(r)}$	$H^2\phi_a^{\mathrm{F}}$
					$\vdots$	$\vdots$	$\vdots$	$\vdots$
					$H^{r-3}\phi_a^{(r-2)}$	$H^{r-3}\phi_a^{(r-1)}$	$H^{r-3}\phi_a^{(r)}$	$H^{r-3}\phi_a^{\mathrm{F}}$
						$H^{r-2}\phi_a^{(r-1)}$	$H^{r-2}\phi_a^{(r)}$	$H^{r-2}\phi_a^{\mathrm{F}}$
							$H^{r-1}\phi_a^{(r)}$	$H^{r-1}\phi_a^{\mathrm{F}}$

$\phi_a^{(i)}$ denotes the ath member of the ith class of second-class primary constraint. $a = 1, 2, \dots, n_i$. n_i may vanish.
ϕ_a^{F} denotes the ath first-class primary constraint. $a = 1, 2, \dots, n_{\mathrm{F}}$.

Proof. This is given in the appendix.

The significance of the division of the second-class primary constraints into r sets will become clear on reading the proof. For the sequel it is only necessary to appreciate that quantities of the form $H^s\phi_a^F$ can be written as linear combinations of first-class constraints and include every first-class constraint.

The motion is generated by

$$H + u_a \cdot \phi_a^F.$$

In this and subsequent expressions summation is over values of lower indices compatible with those of upper ones—in this case $1 \to n_F$. The u_a are arbitrary functions.

3. The generators of surfaces of physically equivalent points

Two points are physically equivalent (PE) if they can be reached in the same time from a common ancestral point. Suppose $x(s)$ is a curve of points PE to some point $x = x(0)$. Then there holds the following:

Theorem 2. For any function X on phase space (which may be a coordinate function)

$$\frac{d}{ds} X(x(s))|_{s=0} = \sum_{a,i} C_a^i \cdot ((H^i\phi_a^F)X)|_x$$

for some set of coefficients C_a^i. Conversely any choice of coefficients defines a curve in the surface PE to x.

Remark. In other words, the first-class constraints generate the PE surfaces.

Proof. An arbitrary curve in the surface PE to x may, at least locally, be constructed as follows. Choose a family of analytic arbitrary functions $u_a(s)$ parametrised by s so that $u_a(0) = 0$. Define

$$H(s) = H + u_a(s) \cdot \phi_a^F.$$

Proceed backwards at time t down the trajectory through x generated by $H(s)$ to reach a point $y(s, t)$ (see figure 1). Then move forwards a time t along the trajectory through $y(s, t)$ generated by H, to reach a point $x(s, t)$. The curve of points $x(s, t)$,

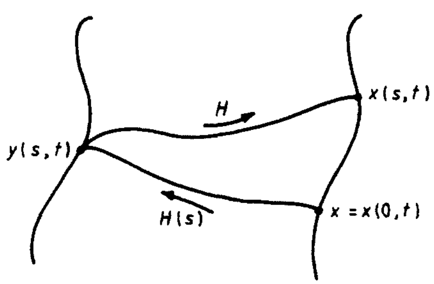

Figure 1.

considered as a function of s for fixed t, is the desired PE curve. By assumption (3) every PE curve can be generated in this way.

By assumption (1), for sufficiently small t,

$$\dot{X} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial s} X(x(s,t))\right)\bigg|_{s=0}$$
$$= \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \frac{t^j t^i}{j!\,i!} (-1)^i (H^k \dot{\phi} H^{i+j-k-1} X)|_x$$
$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{p=0}^{k} \frac{t^{j+1} t^i}{(j+1)!\,i!} (-1)^i \binom{k}{p} ((H^p \dot{\phi}) H^{i+j-p} X)|_x$$

where

$$\dot{\phi} \stackrel{\text{def}}{=} \frac{\partial}{\partial s} H(s)\bigg|_{s=0} = \dot{u}_a(0) \cdot \phi_a.$$

I have assumed that the u are time-independent. This leads to no loss of generality provided none of the trajectories is closed—i.e. provided t is small enough.

It is convenient to make the following successive redefinitions of the indices and their ranges:

$$\left.\begin{array}{ll} m=i+j: & 0\to\infty \\ i & :0\to m \\ k & :0\to m-i \\ p & :0\to k \end{array}\right\} \to \left.\begin{array}{ll} m: & 0\to\infty \\ p: & 0\to m \\ i: & 0\to m-p \\ k: & p\to m-i \end{array}\right\} \to \begin{array}{ll} m & :0\to\infty \\ u=m-p & :0\to m \\ i & :0\to u \\ v=m-i-k & :0\to u-i \end{array}$$

(see figure 2). Then

$$\dot{X}(t) = \sum_{m=0}^{\infty} \sum_{u=0}^{m} \sum_{i=0}^{u} \sum_{v=0}^{u-i} \frac{t^{m+1}}{i!\,(m+1-i)!} (-1)^i \binom{m-i-v}{m-u} ((H^{m-u} \dot{\phi}) H^u X)|_x.$$

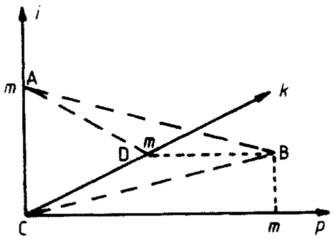

Figure 2.

This simplifies with the help of the results

$$\sum_{s=0}^{u-i} \binom{m-u+s}{m-u} = \binom{m-i+1}{m-u+1}$$

and

$$\sum_{i=0}^{u} \binom{u}{i} (-1)^i = \delta_{u0},$$

the first of which follows from the identity

$$\sum_{s=0}^{u-i}(x+1)^{m-u+s}=\frac{1}{x}[(x+1)^{m-i+1}-(x+1)^{m-u}],$$

to

$$\dot{X}(t)=\sum_{m=0}^{\infty}\frac{t^{m+1}}{(m+1)!}((H^m\dot{\phi})X)|_x.$$

Use of

$$H^m\dot{\phi}=H^m(\dot{u}_a\phi_a^{\mathrm{F}})=\sum_{i=0}^{m}\binom{m}{i}(H^i\dot{u}_a)\cdot(H^{m-i}\phi_a^{\mathrm{F}})$$

then leads to

$$\frac{\partial}{\partial t}\dot{X}(t)=\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\frac{t^m}{m!}\frac{t^i}{i!}(H^i\dot{u}_a)\cdot((H^m\phi_a^{\mathrm{F}})X)|_x$$

(since $H^m\phi_a^{\mathrm{F}}$ vanishes on the physically allowed surface).

Now $\dot{u}_a(0)$ is an arbitrary function of position. Hence

$$f_a(t)\overset{\text{def}}{=}\sum_{i=0}^{\infty}\frac{t^i}{i!}(H^i\dot{u}_a)|_x$$

is, being the value of $\dot{u}_a(0)$ at $y(0,t)$, an arbtriary function of time.

Since the polynomials $\tau^m/m!$ provide a (non-orthonormal) basis for functions on the interval $[0,t]$, it follows that the numbers

$$C_a^m(t)=\int_0^t \mathrm{d}\tau f_a(\tau)\frac{\tau^m}{m!}$$

are arbitrary. But

$$\dot{X}(t)=\sum_{a,m}C_a^m(t)\cdot((H^m\phi_a^{\mathrm{F}})X)|_x. \qquad \text{QED}$$

Appendix. Proof of theorem 1

Consistency is ensured by an iterative process. Proof will be by induction on the sth step of this process. Consider the first step. We demand

$$H_0\phi_a=(\phi_a\phi_b)\cdot u_b \tag{A1}$$

on $\mathcal{R}_0$, $\mathcal{R}_0$ being the physically allowed surface on which the constraints vanish.

The matrix $\phi_a\phi_b$ is antisymmetric. Consequently a suitable linear transformation on the set of constraints will lead to new primary constraints ϕ'_a for which

$$\phi'_a\phi'_b=\left(\begin{array}{c|c}\text{non-singular}&\\ \text{sub-matrix}&0\\ \hline 0&0\end{array}\right)\qquad\text{on }\mathcal{R}_0.$$

Denote by n_1 the number of rows of the non-singular sub-matrix. By assumption (2) n_1 is constant on $\mathcal{R}_0$.

Define

$$\phi_a^{(1)} = \phi'_a \qquad a = 1, \ldots, n_1$$

$$\phi_a^{(+1)} = \phi'_{a+n_1} \qquad a = 1, \ldots, n - n_1$$

$$u_a^{(1)} = u_a \qquad a = 1, \ldots, n_1$$

$$u_a^{(+1)} = u_{a+n_1} \qquad a = 1, \ldots n - n_1.$$

Equations (A1) may be solved for the $u_a^{(1)}$ which are therefore fixed functions. The $u_a^{(+1)}$ remain arbitrary. The equations of motion are genereated by

$$H_1 + u_a^{(+1)} \cdot \phi_a^{(+1)} \stackrel{\text{def}}{=} (H_0 + u_a^{(1)} \cdot \phi_a^{(1)}) + u_a^{(+1)} \cdot \phi_a^{(+1)}.$$

The equations (A1) are only consistent if

$$H_1 \phi_a^{(+1)} (= H_0 \phi_a^{(+1)} \text{ on } \mathscr{R}_0) = 0.$$

These conditions, to the extent that they are independent of each other and the primary constraints, must be imposed as new secondary constraints. There is no loss of generality in treating them as if they were all so independent. It is now necessary to demand that the $H_1\phi_a^{(+1)}$ are conserved. And so on.

Now assume that the following inductive hypotheses hold after the rth iteration:

(a_r) The original, primary, constraints can be written as linear combinations of the members of $r+1$ sets of constraints. The sth set, $s \leqslant r$, contains $n_s \geqslant 0$ elements, and its ath element may be written $\phi_a^{(s)}$. The (r + 1)th set contains $n - \Sigma_{s=1}^r n_s \geqslant 0$ elements, and its ath element may be written $\phi_a^{(+r)}$.

(b_r) The Hamiltonian H_r may be written

$$H_r = H_0 + \sum_{s=1}^{r} \sum_{a=1}^{n_s} u_a^{(s)} \cdot \phi_a^{(s)}$$

where the $u_a^{(s)}$ are not arbitrary.

(c_r) The equations of motion are generated by

$$H_r + u_a^{(+r)} \cdot \phi_a^{(+r)}$$

where the $u_a^{(+r)}$ are arbitrary.

(d_r) Further, secondary, constraints have the form

$$H_r^s \phi_a^{(t)} \qquad \text{for } 1 \leqslant s < t \leqslant +r.$$

(In this and other inequalities $+r$ is to be read as $r+1$. The case $+r$ is always included unless otherwise stated.)

These are possibly not all independent, of each other, or of the primaries. Some may even vanish identically. The primary constraints, and those secondaries with $s \leqslant i$, define the surface $\mathscr{R}_i$. Equality on $\mathscr{R}_i$ will be written $=_i$.

(e_r) The matrix whose elements are, for given s, $\phi_a^{(s+1)} H_r^s \phi_b^{(s+1)}$ is non-singular and (anti-) symmetric on $\mathscr{R}_r$ provided s is odd (even) and less than r.

(f_r) For all functions X,

$$X =_s 0 \Rightarrow H_r X =_{s+1} 0 \qquad \text{if } s < r.$$

(g_r) $\qquad (H_r^s \phi_a^{(i)})(H_r^t \phi_b^{(j)}) =_{s+t} 0 \qquad \text{if } i \text{ or } j > s+t+1 \leqslant r.$

(h_r) $\qquad H_r^s \phi_a^{(t)} =_{s-1} 0 \qquad \text{if } 0 < t \leqslant s \leqslant r.$

From these statements follow:

(i$_r$) $\quad (H_r^s\phi_a^{(i)})(H_r^t\phi_b^{(j)}) =_{s+t} -(H_r^{s-1}\phi_a^{(i)})(H_r^{t+1}\phi_b^{(j)}) \qquad$ if i or $j > s+t \leqslant r$

(a consequence of Jacobi's identity, (f$_r$) and (g$_r$)).

(ii$_r$) $$H_r^s(X\cdot\phi) = X\cdot(H_r^s\phi) + \sum_{p=0}^{s-1}\binom{s}{p}(H_r^{s-p}X)\cdot(H^p\phi)$$

(consequently $H_r^s(X\cdot\phi) =_{s-1} X\cdot(H_r^s\phi)$) where X is any function and ϕ any primary constraint.

(iii$_r$) $\quad X =_s 0 \Rightarrow (H_r^i\phi_a^{(j)})X =_{i+s} 0 \qquad$ if $j > i+s+1 \leqslant r$.

In particular $\phi_a^{(+r)}X =_s 0$ if $s < r$ (a consequence of (g$_r$) and the definition of $\mathscr{R}_s$).

All these statements hold for the case $r=1$. Assume for r and consider the case $r+1$:

Observe that the Poisson bracket (PB) of $H_r + u_a^{(+r)}\cdot\phi_a^{(+r)}$ with every constraint but the $H_r^r\phi_a^{(+r)}$ vanishes on $\mathscr{R}_r$ (by (iii$_r$) and (h$_r$)). The consistency requirement then is

$$H_r^{r+1}\phi_a^{(+r)} = ((H_r^r\phi_a^{(+r)})\phi_b^{(+r)})\cdot u_b^{(+r)}. \tag{A2}$$

Repeated application of (i$_r$) shows that the matrix is (anti-) symmetric if r is odd (even). Hence, using (ii$_r$) and (iii$_r$), there exists an orthogonal matrix A such that equations (A2) are equivalent, on $\mathscr{R}_r$, to

$$H_r^{r+1}\phi_a^{(+r)'} = ((H_r^r\phi_a^{(+r)'})\phi_b^{(+r)'})\cdot u_b^{(+r)'} \tag{A2$'$}$$

where

$$\phi_a^{(+r)'} = A_{a\alpha}\cdot\phi_\alpha^{(+r)} \qquad u_a^{(+r)'} = A_{a\alpha}\cdot u_\alpha^{(+r)}$$

and

$$((H_r^r\phi_a^{(+r)'})\phi_b^{(+r)'}) =_r \left(\begin{array}{c|c} \begin{array}{c}\text{non-singular}\\ \text{sub-matrix}\end{array} & 0 \\ \hline 0 & 0\end{array}\right). \tag{A3}$$

The non-singular sub-matrix is $n_{r+1}\times n_{r+1}$. By assumption (2) n_{r+1} is constant on $\mathscr{R}_r$.

Define

$$\phi_a^{(r+1)} = \phi_a^{(+r)'} \qquad a = 1,\ldots,n_{r+1}$$

and

$$\phi_a^{(+(r+1))} = \phi_{a+n_{r+1}}^{(+r)'} \qquad a = 1,\ldots,n-\sum_{s=1}^{r+1} n_s$$

and similarly for $u_a^{(r+1)}$ and $u_a^{(+(r+1))}$.

The $u_a^{(r+1)}$ are fixed by equation (A2$'$). The $u_a^{+(r+1)}$ remain arbitrary; corresponding to them we have the new secondary constraint equations

$$H_r^{r+1}\phi_a^{+(r+1)} = 0. \tag{A4}$$

Define

$$H_{r+1} = H_r + u_a^{(r+1)}\cdot\phi_a^{(r+1)}.$$

Now consider the inductive claims (a$_{r+1}$) $\rightarrow$ (h$_{r+1}$).

The first three are immediate.

$$(\mathrm{d}_{r+1}) \qquad H^s_{r+1}\phi^{(i)}_a =_{s-1} H^s_r\phi^{(i)}_a \qquad \text{if } i \leqslant +(r+1), \quad s \leqslant r. \tag{A5}$$

[In fact, by (iii$_r$), it is true for $s=1$. Assume for $s<r$. Then

$$H^{s+1}_{r+1}\phi^{(i)}_a =_s (H_r + u^{(r+1)}_b \cdot \phi^{(r+1)}_b)H^s_r\phi^{(i)}_a \qquad \text{by } (\mathrm{f}_r) \text{ and } (\mathrm{iii}_r)$$
$$=_s H^{s+1}_r \phi^{(i)}_a \qquad \text{by } (\mathrm{iii}_r)$$

(since $H^s_r\phi^{(i)}_a =_s 0$ by (d$_r$), (h$_r$) and (ii$_r$)).]

It then readily follows that

$$z \in \mathscr{R}_{s_0} \qquad \text{iff } H^s_{r+1}\phi^{(i)}_a|_z = 0 \qquad \text{for all } s \leqslant s_0 \leqslant r+1, \quad i \leqslant +(r+1).$$

So we may replace H_r by H_{r+1} everywhere in the table of constraints. Observe that (A4) gives a new row.

(e$_{r+1}$) Consequence of (e$_r$), (A3), (A5) and (iii$_r$).

(f$_{r+1}$) Consequence of (f$_r$), (iii$_r$) and the definitions.

(g$_{r+1}$) First observe that, as a consequence of (ii$_r$), (g$_r$) continues to hold if we write $\phi^{(r+1)}_a$ and $\phi^{+(r+1)}_a$ in place of $\phi^{(+r)}_a$ Proof is now by induction. First I shall show

$$\phi^{(i)}_a H^s_{r+1}\phi^{(j)}_b =_s 0 \qquad \text{if } i > s+1 \leqslant r+1.$$

In fact,

$$H^s_{r+1}\phi^{(j)}_b =_{s-1} H^s_r\phi^{(j)}_b \qquad \text{by (A5)}.$$

The result then follows from (iii$_r$) if $s<r$. If $s=r$ and $j \leqslant r$ it follows from (h$_r$) and (iii$_r$). If $s=r$ and $j=(r+1)$ or $+(r+1)$ it follows from (A3). This covers every case.

Now make the inductive hypothesis

$$(H^t_{r+1}\phi^{(i)}_a)(H^s_{r+1}\phi^{(j)}_b) =_{t+s} 0 \qquad \text{if } i > t+s+1 \leqslant r+1 \text{ and } t \leqslant t_0 < r.$$

The same result then follows for t_0+1 by use of Jacobi's identity and (f$_{r+1}$).

$$(\mathrm{h}_{r+1}) \qquad H^i_{r+1}\phi^{(j)}_a =_{i-1} 0 \qquad j \leqslant i \leqslant r+1;$$

if $j \leqslant i \leqslant r$ has a consequence of (A5) and (h$_r$); if $j \leqslant r$ and $i=r+1$ by (A5), (h$_r$) and (f$_{r+1}$); and if $j=i=r+1$ by (A5), (f$_{r+1}$) and the definition of H_{r+1}.

This concludes the inductive part of the proof.

Now for some r, $\mathscr{R}_r = \mathscr{R}_{r-1}$ (it may of course happen that $\mathscr{R}_r$ is empty). for this r define

$$\phi^{\mathrm{F}}_a = \phi^{(+r)}_a \qquad H = H_r \qquad u_a = u^{(+r)}_a.$$

Observe that H is first-class. The ϕ^{F}_a are first-class by (g$_r$). The $H^s\phi^{\mathrm{F}}_a$ are then first-class by the result of Dirac (1964), namely that the PB of first-class quantities is itself first-class.

All that remains is to show that there are no more first-class constraints (and hence that the $H^t\phi^{(s)}_a$ constitute the complete linearly independent set of second-class constraints). This follows if the matrix whose element on the ath row and bth column has the form

$$(H^s\phi^{(i)}_a)(H^t\phi^{(j)}_b) \qquad 0 \leqslant s, t < i, j \text{ respectively} \leqslant r, \quad a, b \leqslant n_i, n_j \text{ respectively}$$

is non-singular on $\mathscr{R}_{r-1}$.

To prove this, it is convenient to re-label the constraints

$$R^i_{ja} \overset{\text{def}}{=} H^i \phi_a^{(r-j)}$$

$$D^i_{ja} \overset{\text{def}}{=} H^{r-i-j-1} \phi_a^{(r-j)} \qquad i = 0, 1, \ldots, \left[\frac{r-2}{2}\right], \quad j = 0, 1, \ldots, r-2(i+1)$$

$$C^i_a \overset{\text{def}}{=} H^i \phi_a^{(2i+1)} \qquad i = 0, \ldots, \left[\frac{r-1}{2}\right]$$

where $[x]$ means greatest lower integral bound on x.

The following results are consequences of (e_r), (g_r) and (i_r):

(i) $R^i_{ja} R^k_{lb} =_{r-1} 0$ for all i, j, k, l, a, b.

(ii) $R^i_{ja} C^k_b =_{r-1} 0$ for all i, j, k, a, b.

(iii) $R^i_{ja} D^k_{lb} =_{r-1} 0$ if $i < k$ or $i = k$ and $j < l$.

(iv) $C^i_a C^j_b =_{r-1} 0$ if $i \neq j$.

(v) The sub-matrices whose elements are $M_{ab} = R^i_{ja} D^i_{jb}$ are non-singular for every i and j.

(vi) The sub-matrices with elements $M_{ab} = C^i_a C^i_b$ are likewise non-singular for every i.

Consequently, the matrix in question has the form given in table A1. Observe that the R–D sub-matrices are block triangular with non-singular blocks on the diagonal.

Table A1.

	R^0	R^1	$\cdots$	C	D^0	D^1	$\cdots$
R^0	0	0	0	0	$\begin{smallmatrix}\times & & 0\\ & \ddots & \\ & & \times\end{smallmatrix}$	0	0
R^1	0	0	0	0		$\begin{smallmatrix}\times & & 0\\ & \ddots & \\ & & \times\end{smallmatrix}$	0
$\vdots$	0	0	0	0			$\cdots$
C	0	0	0	$\begin{smallmatrix}\times & & 0\\ & \ddots & \\ 0 & & \times\end{smallmatrix}$			
D^0	$\begin{smallmatrix}\times & & \\ & \ddots & \\ 0 & & \times\end{smallmatrix}$						
D^1	0	$\begin{smallmatrix}\times & & \\ & \ddots & \\ 0 & & \times\end{smallmatrix}$					
$\vdots$	0	0	$\cdots$				

$\times$ denotes a non-singular submatrix

They are therefore non-singular. The same is true for the C–C sub-matrix. But these sub-matrices are themselves the blocks on the diagonal of the whole matrix which is also block triangular. Therefore the whole matrix is non-singular. QED

Reference

Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science, Yeshiva University) p 87